Quantitative Literacy:
Thinking Between the Lines

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Chapter 8:
Voting and Social Choice

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Lesson Plan

- Measuring voting power: Does my vote count?

- Voting systems: How do we choose a winner?

- Fair division: What is a fair share?

- Apportionment: Am I represented?

Learning Objectives:

▸ Show there is no perfect voting system if there are three or more candidates

▸ Touch on some of the different voting systems used

▸ Understand concepts such as:

  ▸ Instant Runoff Voting

  ▸ The Spoiler Effect

  ▸ Plurality Casting

  ▸ Condorcet Winner

  ▸ Independence of Irrelevant Alternatives

▸ **Voting System:** A set of rules under which a winner in an election is determined.

▸ **Plurality Voting:** The system of voting in which the candidate that receives more votes than any other candidate is the winner.

▸ **Example:** Of four candidates and 100 votes, what is the smallest number of votes needed to win?

▸ **Solution:** If the four candidates have an equal number of votes, they would have $100/4 = 25$ each. So a candidate could have a plurality with as few as 26 votes.

▸ **Spoiler:** A candidate who has no realistic chance of winning but whose presence in the election affects the outcome.

▸ **Example:** In the 2000 election, the Florida vote tally for the candidates was as follows:

Candidate	Votes
George W. Bush	2,912,790
Al Gore	2,912,253
Ralph Nader	97,488
Others	40,579

"Others" includes five other candidates and write-in votes.

▸ **Example (cont.):**

1. What percent voted for Bush? Did anyone achieve majority?

2. Suppose the election only included the top three candidates holding their votes but the votes for "others" were additionally distributed to these three. Could anyone achieve majority?

3. Assume Nader did not participate, "Others" remain, and all voters for Nader are redistributed or don't vote. How many more of Nader's votes must Gore obtain than Bush to win?

4. Exit polls indicate that 21% of Nader voters would have voted for Bush and 47% of them would have voted for Gore. It is estimated that 14 – 17% of Nader votes would have gone to Bush and 32 – 40% would have gone to Gore. Does this indicate that Nader was a spoiler?

▸ **Solution:**

1. Bush received 2,912,790 of the 5,963,110 total votes, which is 2,912,790/5,963,110 or 48.8%, which is not a majority.

2. 5,963,790/2 = 2,981,555, so a majority is 2,981,556 votes. That is 68,766 votes more than were cast for Bush.

3. Bush received 537 more votes than Gore. So Gore would need 538 more Nader votes than Bush.

4. 21% of the Nader votes is 20,472 and 32% is 31,196.

   This gives Bush a total of **2,933,262** and Gore a total of 2,943,449 votes.

   If Nader had not been on the ballot, the outcome of the election would have been different.

   Therefore Nader could have reasonably been considered to be a spoiler for the election.

8.2 Voting systems: How do we choose a winner?

- **Preferential Voting System:** Systems in which voters express their ranked preferences between various candidates, usually with a **ranked ballot** which is used to avoid several rounds of voting and the voter lists his or her candidate preferences. Two examples follow.

- **Top-Two Runoff System:** If no candidate receives majority, there is a new election with only the two highest vote-getters.

- **Elimination Runoff System:** If no candidate receives majority, the lowest vote-getter is eliminated and a vote is taken again among those who are left. This repeats until a majority is reached.

▸ **Example:** Consider the following ranked ballot outcome for 10 voters choosing among three candidates:

Rank	4 voters	4 voters	2 voters
First Choice	Alfred	Gabby	Betty
Second Choice	Betty	Alfred	Gabby
Third Choice	Gabby	Betty	Alfred

▸ Determine the winner under the elimination runoff system.

▸ **Solution:**

  ▸ No candidate has first-choice majority, Betty has the least so she is eliminated from the first round.

Rank	4 voters	4 voters	2 voters
First Choice	Alfred	Gabby	~~Betty~~
Second Choice	~~Betty~~	Alfred	Gabby
Third Choice	Gabby	~~Betty~~	Alfred

  ▸ With Betty eliminated, the table is now as follows:

Rank	4 voters	4 voters	2 voters
First Choice	Alfred	Gabby	Gabby
Second Choice	Gabby	Alfred	Alfred

  ▸ Now the first-choice votes are 4 for Alfred and $4 + 2 = 6$ for Gabby. In this runoff, Gabby has majority and is the winner.

‣ **Borda count:** A method of ranked balloting that assigns for each ballot: 0 points to the choice ranked last, 1 point to next higher choice, and so on. The **Borda winner** is the candidate with the highest Borda count.

‣ **Example:** To decide on food, five friends mark ranked ballots by preference, using the table:

	Pizza	Tacos	Burgers
Ballot 1	2	1	0
Ballot 2	2	1	0
Ballot 3	2	1	0
Ballot 4	0	2	1
Ballot 5	0	2	1

1. Did one of the foods receive a majority for first-choice?
2. Use the Borda count to determine which food should be ordered.

▸ **Solution:**

1. First-place votes are indicated by the number 2. There were three first-place votes for pizza, which is a majority of the five first-place votes.

2. The Borda count is:

   For pizza: $2 + 2 + 2 + 0 + 0 = 6$

   For tacos: $1 + 1 + 1 + 2 + 2 = 7$

   For burgers: $1 + 1 = 2$

   According to the Borda count, the group should order tacos. This is true in spite of pizza receiving majority.

   This lends to our understanding that no voting system is perfect with three or more candidates.

▸ **Example:** The five finalists for the 2009 Heisman Trophy follow:

Player	1st-Place Votes	2nd-Place Votes	3rd-Place Votes
T. Gerhart (Stanford)	222	225	160
M. Ingram (Alabama)	227	236	151
C. McCoy (Texas)	203	188	160
N. Suh (Nebraska)	161	105	122
T. Tebow (Florida)	43	70	121

▸ Determine the Borda counts and the Borda winner.

▸ **Solution:**

Player	1st-Place Votes	2nd-Place Votes	3rd-Place Votes
T. Gerhart (Stanford)	222	225	160
M. Ingram (Alabama)	227	236	151
C. McCoy (Texas)	203	188	160
N. Suh (Nebraska)	161	105	122
T. Tebow (Florida)	43	70	121
Borda count value	**2**	**1**	**0**

Borda count for T. Gerhart $= 222 \times 2 + 225 \times 1 + 160 \times 0 = 669$

M. Ingram $= 227 \times 2 + 236 \times 1 + 151 \times 0 = $ **690**

C. McCoy $= 203 \times 2 + 188 \times 1 + 160 \times 0 = 594$

N. Suh $= 161 \times 2 + 105 \times 1 + 122 \times 0 = 427$

T. Tebow $= 43 \times 2 + 70 \times 1 + 121 \times 0 = 156$

▸ **Example:** Consider the following ranked ballot outcome for 100 voters choosing from options A, B, C, D:

Rank	28 votes	25 votes	24 votes	23 votes
1st Choice	A	B	C	D
2nd Choice	D	C	D	C
3rd Choice	B	D	B	B
4th Choice	C	A	A	A

1. Who wins under plurality voting?
2. Who wins under the top-two runoff system?
3. Who wins under the elimination runoff system?
4. Who wins under the Borda count system?

▶ **Solution:**

1. Under plurality voting only first-choice picks are considered. In that case, candidate A has the most first- choice votes with 28/100.

2. For a runoff with only the top two candidates, A and B, C and D are eliminated from the table, as below:

Rank	28 Votes	25 Votes	24 Votes	23 Votes
Adjusted 1st Choice	A	B	B	B
Adjusted 2nd Choice	B	A	A	A

▶ In this runoff, B has 72 first-choice votes; this is clearly a majority, so B is the winner.

▸ **Solution (cont.):**

3. In the first round of elimination runoff, D has the fewest votes and is eliminated, the table now follows:

Rank	28 votes	25 votes	24 votes	23 votes
1st Choice	A	B	C	C
2nd Choice	B	C	B	B
3rd Choice	C	A	A	A

▸ Now B has the fewest votes and is eliminated:

Rank	28 votes	25 votes	24 votes	23 votes
1st Choice	A	C	C	C
2nd Choice	C	A	A	A

▸ C wins in this runoff by a majority of 72 votes.

▸ **Solution (cont.):**

4. The Borda count incorporated into the original table:

Rank	Borda Value	28 votes	25 votes	24 votes	23 votes
1st Choice	**3**	A	B	C	D
2nd Choice	**2**	D	C	D	C
3rd Choice	**1**	B	D	B	B
4th Choice	**0**	C	A	A	A

So the Borda count for each candidate is:

$A = 28 \times 3 + 25 \times 0 + 24 \times 0 + 23 \times 0 = 84$

$B = 28 \times 1 + 25 \times 3 + 24 \times 1 + 23 \times 1 = 150$

$C = 28 \times 0 + 25 \times 2 + 24 \times 3 + 23 \times 2 = 168$

$D = 28 \times 2 + 25 \times 1 + 24 \times 2 + 23 \times 3 = 198$

Hence D is the winner based on Borda count.

Common Preferential Systems

▸ **Plurality:** The candidate with the most votes wins.

▸ **Top-two runoff:** If no one garners a majority of the votes, a second election is held with the top two getters as the only candidates.

▸ **Elimination runoff:** Successive elections are held where the candidate with the smallest number of votes is eliminated. This continues until there is a majority winner.

▸ **Borda count:** Voters rank the candidates first to last. The last-place candidate gets 0 points, the next 1 point, and so on. The candidate with the most points wins.

▸ A **Condorcet Winner** is a candidate who beats each of the other candidates in a 1– 1 election.

▸ **Example:** Suppose in an election there are 7 voters and three candidates, A, B, and C; the voters' preferences follow:

Preferences	3 voters	2 voters	2 voters
1st Choice	A	C	C
2nd Choice	B	B	A
3rd Choice	C	A	B

Is there a Condorcet winner? If so, which candidate?

▸ **Solution:**

Find the results of each head-to-head contest:

▸ Consider A and B: there are $3 + 2 = 5$ voters who rank A over B, and only 2 who rank B over A. So A wins versus B.

▸ The results of the other head-to-head contests are:

  ▸ A and C: C wins by 1 (4 to 3)
  ▸ B and C: C wins by 1 (4 to 3)

▸ There is a Condorcet winner—it is candidate C.

8.2 Voting systems: How do we choose a winner?

▸ **Example:** In an election there are seven voters and candidates A, B, C, and D. The tally of ranked ballots follows:

Voter:	1	2	3	4	5	6	7
1st Choice	A	A	B	C	D	A	C
2nd Choice	B	D	A	B	B	D	B
3rd Choice	C	B	C	A	A	B	A
4th Choice	D	C	D	D	C	C	D

1. Who wins the plurality system?
2. Who wins the top-two runoff system?
3. Who wins in the elimination runoff system?
4. Who wins the Borda count?
5. Is there a Condorcet winner? If so, which candidate?

8.2 Voting systems: How do we choose a winner?

▸ **Solution:**

1. In a plurality voting, only first choices are considered. Candidate A has 3 votes, B has 1, and C has 2 votes. A has the most, so A wins.

2. The first-place winner is A. Candidate C is second with 2 votes. In a runoff with A and C, A wins with 5 votes to 2.

3. Because B and D only get one vote each, they are eliminated and A wins.

4. The Borda count is as follows: $A = 3 \times 3 + 1 \times 2 + 3 \times 1 = 14$

$$B = 1 \times 3 + 4 \times 2 + 2 \times 1 = 13$$
$$C = 2 \times 3 + 2 \times 1 = 8$$
$$D = 1 \times 3 + 2 \times 2 = 7$$

So, A wins the Borda count.

5. There is a Condorcet winner. The head-to-head outcomes are as follows:

B beats A (4 to 3)   B beats C (5 to 2)   B beats D (4 to 3)

So, B is the Condorcet winner.

8.2 Voting systems: How do we choose a winner?

▸ The **Condorcet winner criterion** says that if there's a Condorcet winner, then he or she should be the winner of the whole election.

▸ The condition of **Independence of irrelevant alternatives** states: Supposing candidate A wins an election and B loses, and another election follows in which no voter changes their preference concerning A and B, B should still lose to A no matter what happens concerning the other candidates.

▸ **Arrow's Impossibility Theorem:** If there are three or more candidates, there is no voting system (other than dictatorship) for which the Condorcet winner criterion and the Independence of irrelevant alternatives hold.

▶ **Voting power:** does your vote count

- ▶ coalitions, quotas, swing voters
- ▶ Banzhaf power index
- ▶ Stanley-Shubik power index

▶ **Voting systems:** choosing a winner

- ▶ plurality
- ▶ top-two runoff
- ▶ elimination runoff
- ▶ Borda count
- ▶ Condorcet winner

Fair division:

- Methods: divide and choose

  adjusted winner procedure

Apportionment:

- Hamilton's method
- Jefferson's method
- Adjusted divisor methods
- Huntington-Hill method